Non - traditional Approaches to Selected Stereometric Problems through DGS

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Abstract

Mathematics education through dynamic geometry systems (DGS) provides new dimensions for both teaching and learning. The creation of an experimental, interactive and dynamic environment makes possible the teaching of mathematics based on facts, personal experience and, in many cases, on non-traditional approaches to the solution of tasks in geometry.
The paper provides

a) *a non-traditional approach to visualisation of the geometrical principle of regular polyhedron duality through the creation of animated interactive models in Cabri3D;*
b) *Cavalieri´s principle vs. Archimedes's proof via dynamic models in Cabri3D.*

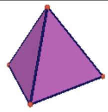
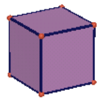
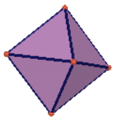
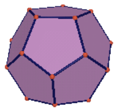
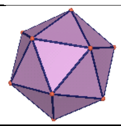
1. Introduction

There is currently a project named Modernization of Primary and Secondary Education in Slovakia. Despite several years of (unorganized) efforts to integrate the ICT elements in the teaching of mathematics in our country, the result has not been satisfactory. Many teachers have adhered to the traditional teaching based mainly on the transmissive model of knowledge delivery. There are two goals in this project: to retrain the primary and secondary school teachers with modernized approaches and to develop educational materials based on ICTs. This paper also discusses dynamic applets necessary for the development of the teaching materials. Two entirely different methods related to the teaching and learning of stereometrics are analyzed and compared.

2. Platonic solids and duality principle

The Platonic solids group includes convex regular polyhedrons that have similar regular polygons on all faces and each vertex is incident with an equal number of edges or faces. In the three dimensional Euclidean space, there are just five types of regular polyhedrons: a regular tetrahedron, cube, regular octahedron, regular dodecahedron and a regular icosahedron. In mathematics education, students at different school levels learn about the basic features of regular polyhedrons. The most important of them is the observation of solids regarding the number of faces, vertices and edges (*Table 1*). Each regular solid can be uniquely determined with the so-called Schläfli symbol. The *Schläfli symbol* is a notation defining the corresponding regular polyhedron. In the ordered pair {*p*, *q*}, *p* represents the number of vertices of each face and *q* represents the number of edges incidental with each vertex.

Table 1 Regular polyhedrons and their attributes

regular solids		number of vertices	number of faces	number of edges	Schläfli symbol
regular tetrahedron		4	4	6	{3, 3}
regular hexahedron (cube)		8	6	12	{4, 3}
regular octahedron		6	8	12	{3, 4}
regular dodecahedron		20	12	30	{5, 3}
regular icosahedron		12	20	30	{3, 5}

The notation of the basic characteristics of regular solids given in the table shows that between some of the pairs the numbers of vertices correspond to the numbers of faces. (The connection is indicated in the table with arrows; it can lead to acquisition of intuitive understanding of the Euler´s sentence.) The given relation can also be noted in the form of the Schläfli symbol, where the reciprocal exchange of the notation order in the characteristics {*p*, *q*} to {*q*, *p*} leads to the description of another regular solid. The relation thus presented is generally known in geometry as the **duality principle**. Thus, it could be stated that the solid {*q*, *p*} is the dual solid to the solid {*p*, *q*} and vice versa. In the geometrical interpretation, all the faces of the regular solid were exchanged by the points (middle points of the polyhedron, or circumscribed circles of the given faces) that become the vertices of the dual regular solid. It is clear that regular octahedrons are dual solids to cubes (Fig. 1a, 1b), regular icosahedrons to regular dodecahedrons and vice versa. Regular tetrahedron is dual solid to itself.

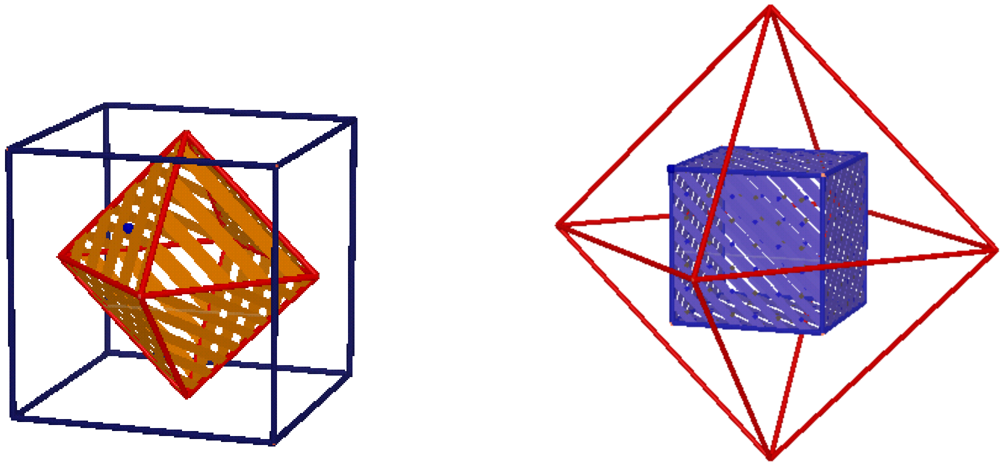


Figure 1a, 1b: *Duality of a cube and a regular octahedron*

2. 1. Construction of dual regular solids in DGS

The theoretical base described above can be readily applied to the construction of dual solids in DGS. Dynamic geometry systems allow the simple construction of a dual solid to a selected regular solid. It is enough to construct the midpoints of some faces (it is not necessary to locate all of them) and then to use the geometrical tools to construct the corresponding dual regular solid (Fig. 2).

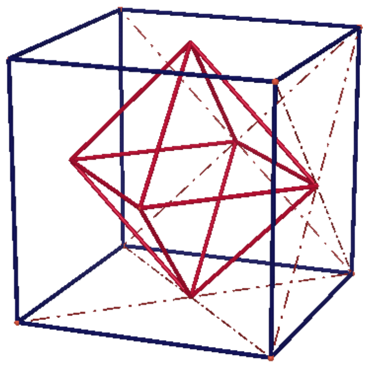


Figure 2: *Traditional construction of dual regular solid*

Because of the dynamic possibilities of DGS (Cabri 3D specifically), it is possible to consider the visualisation of regular solids duality with animation methods. Creating dual models animated with geometry permits the next geometrical relations to be sought, which is a condition for the successful implementation of the construction into the environment of dynamic geometry and entails an important didactic benefit.

The following instructions for creating a dynamic model are provided to illustrate the duality of regular tetrahedrons (Fig. 3a, 3b, 3c).

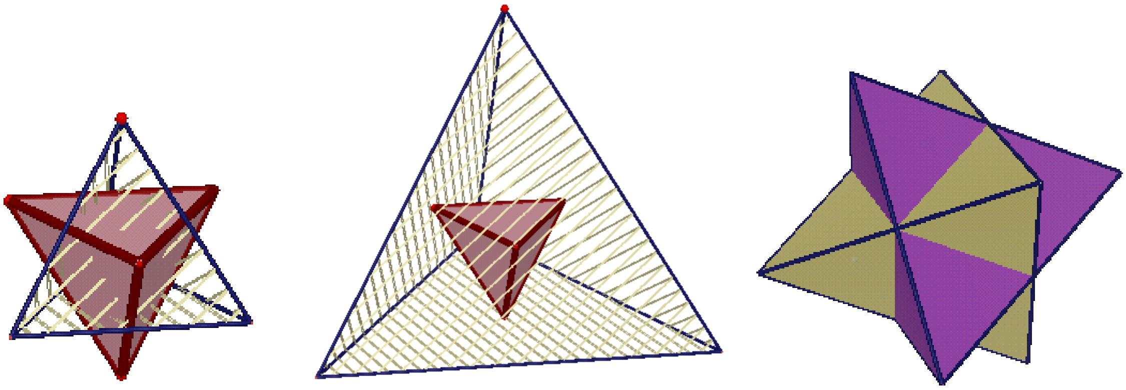


Figure 3a, 3b, 3c: *Dynamic model (stvorsten.cg3) of two mutually dual regular tetrahedrons*

The first steps in the construction are to create three mutually dual tetrahedrons E_1, E_2, E_3 (Fig. 4a, 4b).

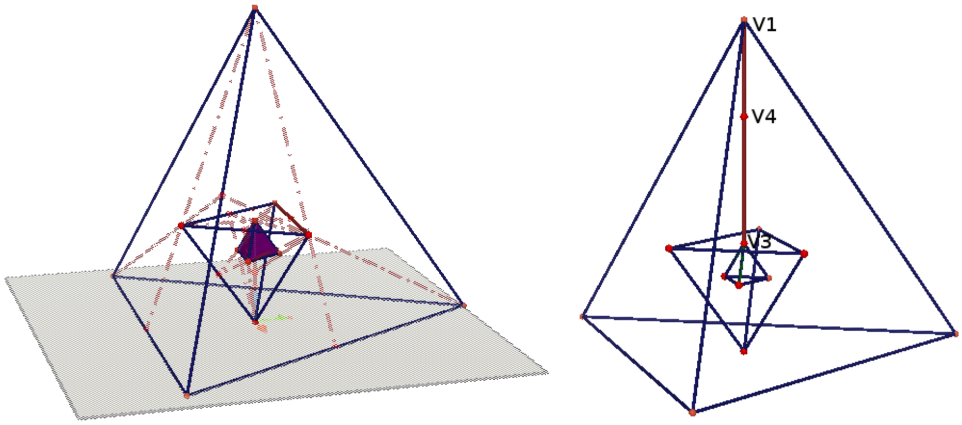


Figure 4a, 4b: *Construction procedure of dynamic model*

The regular tetrahedrons E_1 (the biggest) and E_3 (the smallest) present an example and an image in **homothety** in which centre S is uniquely determined by the pair. It is necessary to construct the next tetrahedron E_4 that will be scaled up and down according to the current position of one of its vertices. The regular tetrahedrons E_1 and E_3 will create the upper and lower limits for the moving tetrahedron E_4. That is why a line segment between the corresponding vertices V_1 and V_3 should be marked; on the segment, the free point V_4 will be constructed as a representative of a vertex of the regular tetrahedron E_4. The regular tetrahedron E_4 will be an image of the tetrahedron in homothety given by the centre S, and a coefficient that can also be defined in Cabri as a ratio of objects of the same type. In our example, the homothetical coefficient will be the ratio of the segments $|SV_4|:|SV_3|$. To complete the virtual model illustrating the duality of a regular tetrahedron all that is required is to define the movement for the point V_4 along the segment V_1V_3 and the result is extremely showy.

An interesting structure is created from two mutually dual regular tetrahedrons in the configuration shown in Fig. 3c. Kepler called the given structure *stella octangula*, i.e. an eight-pointed star. It is interesting that the twelve edges of both regular tetrahedrons create the diagonals of the six faces of the cube. Interactive geometry is an especially well-suited environment for handling the given solids. The intersection of both the regular tetrahedrons in the given composition can also be observed and investigated.

Analogical geometrical relations can also be found creating a dynamic model of cubes and regular octahedrons duality (Fig. 5a, 5b, 5c), or the duality of regular dodecahedrons and a regular icosahedron (Fig. 6a, 6b, 6c). In addition to well-executed and neat models, the most important contribution of dynamic geometry to the given issue is the need to deal with non-standard geometrical relations between the given regular polyhedrons.

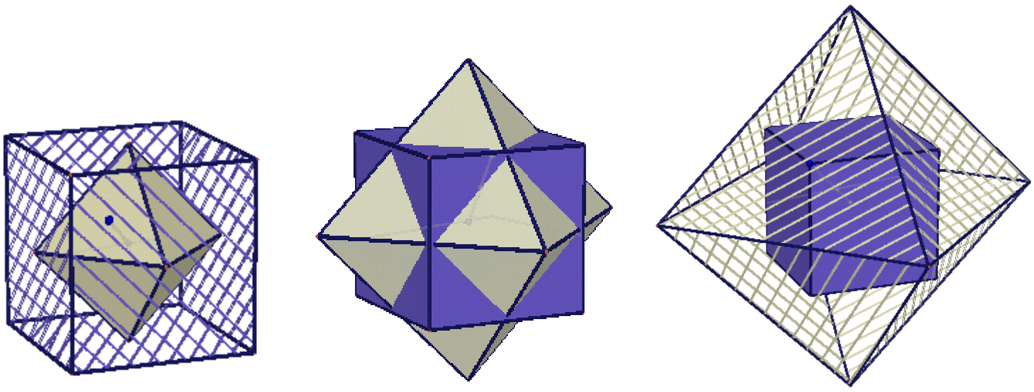


Figure 5a, 5b, 5c: Dynamic model *(dualita_kocka_osemsten.cg3)*
of a cube and dual regular octahedrons

Within the frame of traditional education in the characteristics of regular polyhedrons at basic and secondary schools, or in the education of future teachers of mathematics, one of the highest educational goals is perhaps the ability to show the mutual duality of two regular solids. The construction tools and dynamic possibilities of DG systems allow a more detailed investigation of regular solids or geometrical relations between regular polyhedrons. They also entail an inspirational environment that motivates students to discover the ensuing relations and to use their existing knowledge.

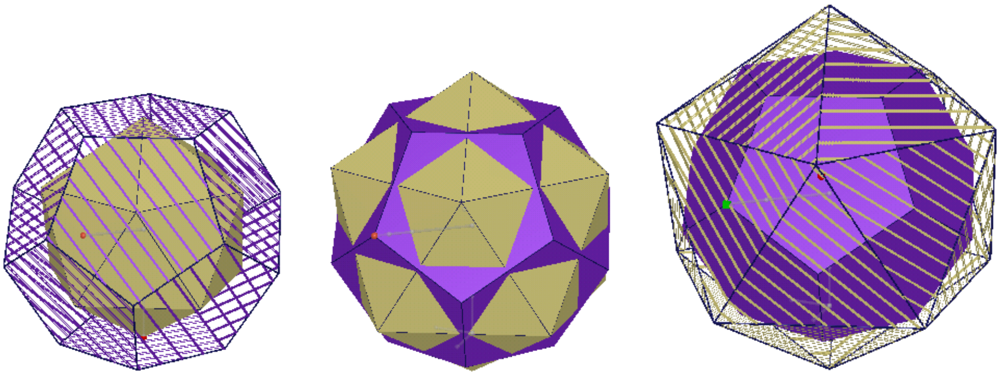


Figure 6a, 6b, 6c: Dynamic models *(dualita_dvanast_sten.cg3)*
of the regular dodecahedron and dual regular icosahedron

2. 2. Suggestions for activities focused on exploring the duality of regular solids

One of the viable methods of studying regular polyhedrons is through examining the number of their vertices, faces, edges, and recording the data in a table (e.g. Table 1). When interpreting the data in the table it would be appropriate to formulate the questions as follows:

- How many faces a cube has? How many vertices a regular tetrahedron has?
- What is the number of vertices of a cube? What is the number of faces of a regular tetrahedron?
- Analogically, examine the relationship between the number of faces and vertices of a regular dodecahedron and a regular icosahedron.
- Identify a solid the vertices of which are the centres of the cube faces.
- Try to make a dynamic construction of this issue using the Cabri 3D. Students can identify all the centres of the cube faces step by step and then plot a convex polyhedron with its vertices being constituted by the centers of the cube faces.
- Using the dynamic geometry tools, verify the regularity of the plotted solid.
- Further exploration may continue with a regular octahedron focusing on the centers of its faces. (As the polygons with an odd number of vertices have no center, we shall replace the given term by a more accurate one, i.e. a "center of a circumscribed circle" or a "circumcenter" of the given polygon.) Then a plot will be drawn of a convex solid the vertices of which are the "centers of the faces of a regular octahedron." And again, regularity of the resulting solid will have to be verified.
- The same approach is to be applied in case of a regular tetrahedron. What solid will we get?

Experience has shown that in case of traditional plotting by the means of paper and pencil, students loose the perception of the solid being plotted. Designing by means of the Cabri3D is clearer and the experience has shown that even less proficient students achieve positive results when using a ruler and a compass through the DGS. Moreover, the dynamic geometry environment also involves the application of students' own creativity, as there are several ways of plotting the solid according to the given requirements, depending on which design tools the student chooses in order to achieve the given task.

A more demanding design of mutually dual solids arises when attempting to visualize their dynamic penetration. In such a case, further geometric knowledge is required. One of the solution methods is shown using the example of a tetrahedron duality (Fig. 3, Fig. 4). An ideal outcome is recorded when students discover the geometric essence of the solution to the given task. One of the ways would be to apply the method of experimenting with dynamic models prepared in advance (in our case the use can be made of a change in the position of the V4 point), by which the principle of enlarging and reducing one of the dual solids can be discovered. The use of homothety and its features in this task is demanding also because it is applied to solving a stereometric task.

3. Cavalieri's Principle

The volume formulas of solids are implemented in qualitative transitions in maths education. These transitions guide the pupils to basic volume theorem.

Volume theorem
If the top and bottom bases of a solid are equal in area, lie in parallel planes, and every section of the solid parallel to the bases is equal in area to that of the base, then the volume of the solid is the product of base and altitude.

The most relevant results of this education are the volume formulas of the solids. A teacher uses a lot of analogy to receive the cone volume formula from the volume theorem. This analogy is a way from prism properties to the volume formula of the cone and disinherits the sphere volume problem.

How to derive the sphere volume formula?

The Cavalieri's principle can give some inspiration. This principle is based on an ancient Greek method of exhaustion.

Cavalieri's Principle
If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal [8].

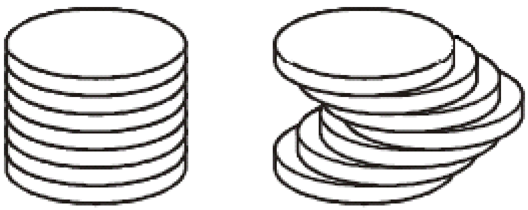


Figure 7: *Cavalieri's principle*

3. 1. Observation via using Cabri 3D

In this section we mention some opportunities how to use Cabri 3D software. It is easy to construct three solids in this software: a cylinder, a cone and a hemisphere. The radii and altitude of the solids can be changed arbitrarily, but it is practical to make the radii and altitude of these solids equal. This condition allows us to compare their volumes.

Let's mark the united radius as *r*. We need to change the parameter r and this is a good occasion for Cabri 3D. The radius r can be implemented in the software as a segment which length can be configured with a manipulation of its ending points. The segment r we use to construct the cylinder, cone and hemisphere. If we measure the volumes of the considered solids then appears the problem task for students.

If we modulate the radius r then the volumes of the solids also change. What value has the ratio $V_{cylinder} : V_{sphere} : V_{cone}$?

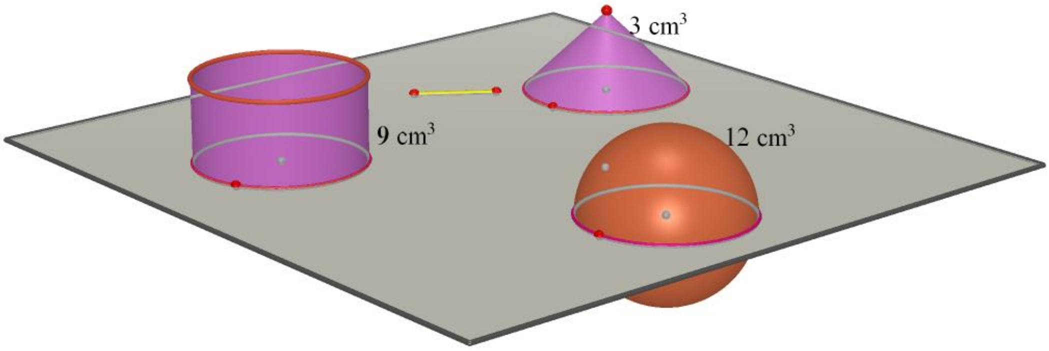


Figure 8: *Cylinder, cone and hemisphere with equal radii and altitude (obr_1.cg3)*

The interaction among the length of the radius r and the computer calculated volumes leads us to hypothesis that ratio is $V_{cylinder} : V_{sphere} : V_{cone} = 3:4:1$. How do we prove it? We have no volume formula of the sphere.

The idea is based on Cavalieri's principle. In software Cabri 3D we realize first experiments. We insert the cone to the cylinder and cut the solids by plane parallel to the base of this cylinder.

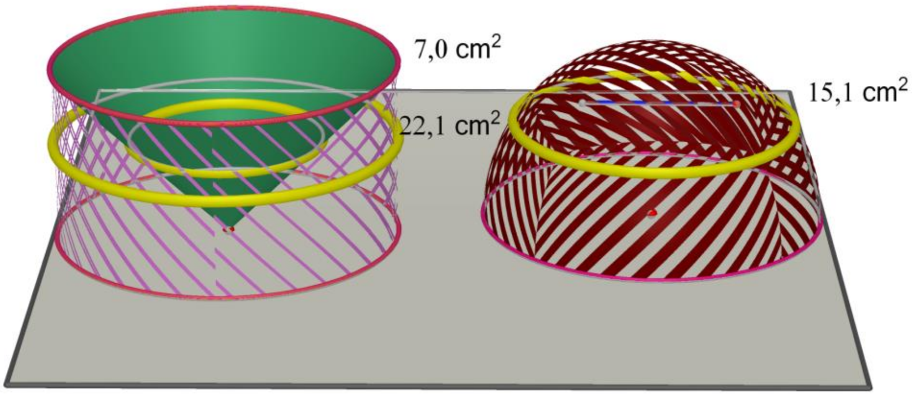


Figure 9: *Cross sections by plane parallel to base – dynamic model (obr_2.cg3)*

The plane cuts the solids in three circles, which areas can be calculated a compared using Cabri 3D utilities. The students can observe that $A_{cylinder} - A_{cone} = A_{sphere}$ ($A_{cylinder}$ means area of the circle which is intersection cylinder and a plane, etc.) and this partial result is real in all situations where we change the radius r.

This circumstantial evidence indicates application of Cavalieri's principle. If the areas of plane sections are equal then the volumes are equal, too.

We need to prove it.

We denote the distance of the planes as h, radius of the cone circle as r_1 and the radius of the hemisphere as ρ. The cone has a base with radius r which is equal to its altitude and that implies $h = r_1$. By using Pythagorean Theorem we obtain $\rho^2 = r^2 - r_1^2$.

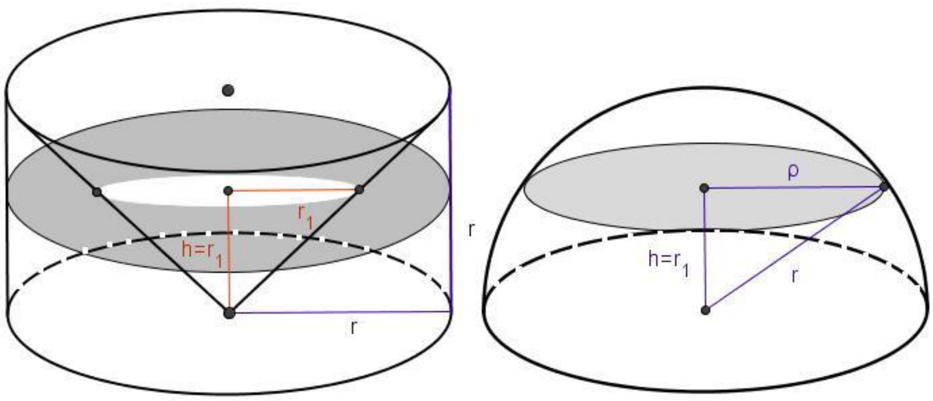


Figure 10: *Details of the cross sections (obr_3.cg3)*

The area of the shape between the two circles is $A_1 = \pi\left(r^2 - r_1^2\right)$ and the area A_2 of the circle in the sphere is equal to $A_2 = \pi\rho^2 = \pi\left(r^2 - r_1^2\right)$. By Cavalieri's Principle (Cavalieris_Principle.cg3) this implies that the volumes of the bodies are equal and hold

$$V_{cylinder} - V_{cone} = \frac{1}{2}V_{sphere}.$$

Precisely

$$2\left(\pi r^3 - \frac{1}{3}\pi r^3\right) = V_{sphere}$$

$$\frac{4}{3}\pi r^3 = V_{sphere}$$

From the last formula results $V_{cylinder} : V_{sphere} : V_{cone} = 3:4:1$, if they have equal radius r and altitude.

This result completes the proof of the hypothesis and also derives the sphere volume formula. From the point of view written above the volume formula of the sphere is derived with connecting link to pieces of knowledge about volume of cylinder and cone.

Remark

The other method of derivation of the sphere volume is known as Archimedes's solution to the sphere volume problem. This approach involves the physics law of the lever. Archimedes used specific ideas about thin layers which cut the solids and then considered about the position of the fulcrum. Archimedes discovered the theorem but did not ground the rigorous proof [3, p. 91-97]. Motivation of Archimedes´ work is illustrated on Fig. 11. There is a sketch of weighing machine which correctly measures the sum of volumes of the sphere and cone and on the other side they are balanced with the volume of the cylinder. The applet is prepared so that the user can change the radii of all solids.

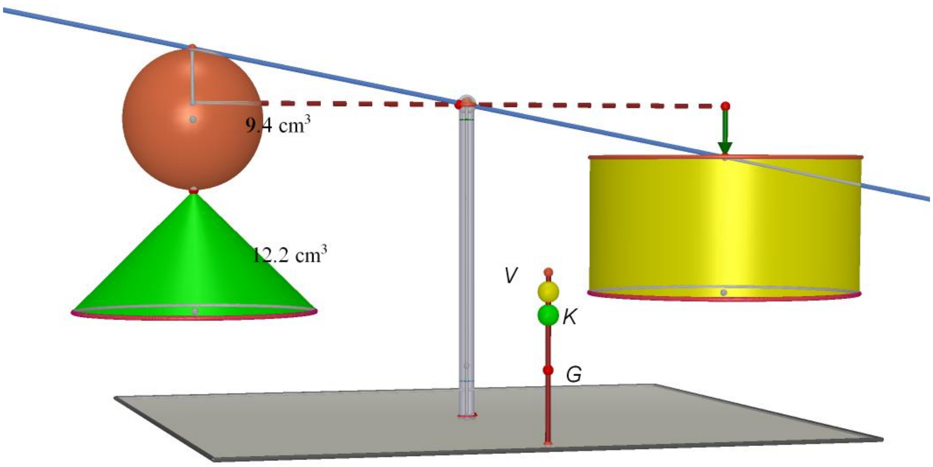


Figure 11: *Motivation - Archimedes's approach based on the lever (Archimedes_lever_1.cg3)*

4. Conclusions

Despite the fact that the general principle of duality is theoretically complicated, it is possible to find and illustrate its basic attributes in a simple form on examples of regular solids duality, even with students in higher secondary education, or during the preparation of future teachers of mathematics at a university level. A new approach to the given problem can be obtained in the geometrical interpretation in DGS. In DGS, animation tools can be used to show the mutual duality of regular solids, using the visualisation process of the penetration of one regular solid into its dual one and conversely. However, the trick assumes the disclosure of further geometrical relations between the mutually dual solids.

On the other side, we have presented the derivation of the sphere volume formula by the Cavalieri´s Principle. The sphere volume formula is formula, which derivation doesn´t explained in standard math textbooks although it is simple. We have used DGS to animate the relationships between the volumes of a cylinder, cone and a hemisphere. The complex idea of Archimedes´s approach based on the lever, which is also famous from the history [3], has been shown for assistance DGS, as well.

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